ERROR PROPAGATION

For sums, differences, products, and quotients, propagation of errors is done as follows. (These formulas can easily be calculated using calculus, using the differential as the associated error of a quantity. From that point of view, the above formulas are simply equivalent to the chain rule. We will see this in more detail later.)

**Rule 1:** (the sum rule) If you are adding or subtracting two numbers $x$ and $y$, each with associated errors $\delta x$ and $\delta y$ respectively, add the errors of the two numbers to get the error associated with the sum or difference. That is,

$$
\delta (x + y) = \delta x + \delta y,
$$

$$
\delta (x - y) = \delta x + \delta y.
$$

**Rule 2:** (the product rule) If you are multiplying or dividing two numbers $x$ and $y$, each with associated errors $\delta x$ and $\delta y$ respectively, add the fractional (or percent) errors of the two numbers, and that will be the fractional (or percent) error of the product or quotient.

$$
\frac{\delta (xy)}{xy} = \frac{\delta x}{x} + \frac{\delta y}{y},
$$

$$
\frac{\delta (x/y)}{x/y} = \frac{\delta x}{x} + \frac{\delta y}{y}.
$$

Note - These formulas can easily be calculated using multivariable calculus and partial derivatives (the stuff of Calc III), associating the differential with the error of a quantity. But we can bypass multivariate calculus for the moment if we think of each of the variables all as functions of some underlying parameter $s$. For example, suppose we consider the derivative of a product of variables $x$ and $y$ as a function of $s$. Then define $f(s) = x(s)y(s)$ and

$$
\frac{df}{ds} = \frac{dx}{ds}y + \frac{dy}{ds}x.
$$

Now “multiply by $ds$” to get the form of the equation in terms of differentials:

$$
df = y\, dx + x\, dy.
$$

Finally divide by $f = xy$ to obtain

$$
\frac{df}{f} = \frac{y\, dx + x\, dy}{xy} = \frac{dx}{x} + \frac{dy}{y},
$$

which is the same formula as the rule for products above, but with the error replaced by the differential. Note that if $f = \log x$ then using the same argument, $df = dx/x$. So this also explains why the error associated with the log of a quantity is the same as the fractional error of the quantity, as we met in lab 1.

Note that if we apply the above to subtraction, we must realize an important subtlety. Suppose we want the error of $f = x - y$. If we treat them both as functions of a parameter $s$ as above, we have

$$
\frac{df}{ds} = \frac{x}{ds} - \frac{dy}{ds},
$$

or

$$
df = dx - dy.
$$
However, because an error is expression the range of knowledge about the quantity, e.g. $x$ is highly likely in the range between $x + \delta x$ and $x - \delta x$, considering the possible range of error for $f$, we would transcribe this into

$$f \pm \delta f = x \pm \delta x - (y \pm \delta y).$$

Now the largest positive deviation for $f$ would be if $\delta x$ were positive and $\delta y$ were negative, and the largest negative deviation is the other way around. So evidently, the highest value in the range for $f$ is

$$f + \delta f = x - y + \delta x + \delta y$$

and the lowest value is

$$f - \delta f = x - y - \delta x - \delta y.$$

This means that we would then need to write

$$f \pm \delta f = (x - y) \pm (\delta x + \delta y).$$

In other words, errors can never cancel each other! They always add up, so we have to add the absolute magnitudes of the respective errors.

In general, if we have a function $f(x, y)$, thinking in terms of $x$ and $y$ both being a function of a parameter $s$, we have

$$\frac{df}{ds} = \frac{df}{dx} \bigg|_y \frac{dx}{ds} + \frac{df}{dy} \bigg|_x \frac{dy}{ds},$$

or

$$df = \frac{df}{dx} \bigg|_y dx + \frac{df}{dy} \bigg|_x dy,$$

where the derivatives are understood to mean that in taking the derivative with respect to $x$, $y$ is held constant, and vice versa. That is, the notation

$$\frac{df}{dx} \bigg|_y$$

means to take the derivative of $f$ with respect to $x$, while holding $y$ constant, and vice versa. In fact, we might as well introduce the Calc III notation here,

$$\frac{df}{dx} \bigg|_y = \frac{\partial f}{\partial x},$$

and similar for $y$. Thus our equation becomes

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

When this is translated into an error, incorporating the absolute magnitude as above because errors always add, we get

$$\delta f = \left| \frac{\partial f}{\partial x} \right| \delta x + \left| \frac{\partial f}{\partial y} \right| \delta y.$$
As an explicit example of error propagation, suppose we are measuring the time it takes for a cart to roll down an inclined plane as in lab 2, and we know the length and the angle of the inclined plane. We then calculate the acceleration using \( x = \frac{1}{2}at^2 \) where \( x \) is the distance traveled and \( t \) is the average values of times for that distance. Using the formula above for propagation of errors for a product, we get in general

\[
\delta a = a \left( \frac{\delta x}{x} + \frac{\delta t}{t} \right)
\]

\[
= \frac{2x}{t^2} \left( \frac{\delta x}{x} + 2 \frac{\delta t}{t} \right)
\]

\[
= \frac{2}{t^2} \delta x + \frac{4x}{t} \delta t.
\]

Another example, relevant to lab 3, is the case of trajectory motion. Suppose we have measured the distance \( x \) that a projectile travels, using several trials and we calculated the average \( \bar{x} \) and the error \( \delta x \) for these trials. Suppose we know the time of flight, by knowing the height (assume we know it fairly precisely, so the associated error for the time will be very small) from which the projectile drops. Then using the product relation above, we could find the standard deviation associated with the velocity derived from \( v = \bar{x}/t \), by using the propagation of products;

\[
\delta v = v \left( \frac{\delta x}{\bar{x}} + \frac{\delta t}{t} \right) = v \frac{\delta x}{\bar{x}},
\]

where the last step follows because we are assuming that \( \delta t \) is so small as to be negligible. Then using \( \bar{x} = vt \) we get

\[
\delta v = \frac{1}{t} \delta x.
\]

However, if we calculated this using differentials, we would get a slightly different answer. Let \( x \) and \( t \) be functions of (the fictitious parameter) \( s \). Then using the quotient rule,

\[
\frac{dv}{ds} = \frac{dx}{ds} \frac{t - x \frac{dt}{ds}}{xt},
\]

or, multiplying by \( ds \),

\[
dv = \frac{tdx - xdt}{xt}.
\]

But if we replace differentials by errors, we are subtracting rather than adding. However, because errors never cancel, we get

\[
\delta v = \frac{t\delta x + \delta dt}{xt}.
\]

**AVERAGE (MEAN) VALUE**

In a typical scientific experiment, measurements will be repeated a number of times, and the result for the measurement is taken to be the average of all measurements taken. Typically a bar over the letter that stands for the quantity in question indicates the average
of that quantity. For a set of measurements for a variable called \( x \) for example, if the measurements have values \( x_1, x_2, x_3, \ldots x_N \), then the average value (\( \bar{x} \)) is given by

\[
\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_N}{N} = \frac{1}{N} \sum_{i=1}^{N} x_i,
\]

where the summation sign \( \Sigma \) is a shorthand notation indicating the sum of \( N \) measurements from \( x_1 \) to \( x_N \). (\( \bar{x} \) is commonly referred to as the average value or mean value of \( x \).)

The error associated with the average can be calculated from the errors of the individual data points using the above rules. More specifically, to get the error associated with the sum of \( N \) values, simply add the errors associated with each individual value together. Now to get the average, you must divide the sum by \( N \). Because \( N \) is exact (there is no error associated with it), the product rule above tells you that the error associated with the average is simply the error associated with the sum divided by \( N \). That is

\[
\delta((x_1 + \cdots + x_N)) = \sum_{i=1}^{N} \delta x_i \quad \text{(rule 1)}
\]

\[
\frac{\delta \bar{x}}{\bar{x}} = \left( \frac{\delta((x_1 + \cdots + x_N))}{(x_1 + \cdots + x_N)} + \frac{\delta N}{N} \right) = \frac{\delta((x_1 + \cdots + x_N))}{(x_1 + \cdots + x_N)}. \quad \text{(rule 2 and } \delta N = 0)\]

Therefore

\[
\delta \bar{x} = \frac{1}{N} (x_1 + \cdots + x_N) \frac{\delta((x_1 + \cdots + x_N))}{(x_1 + \cdots + x_N)} = \frac{1}{N} \sum_{i=1}^{N} \delta x_i.
\]

Note, while this is true for a few unrelated measurements, it is a little strange if we consider that after making lots and lots of measurements, the error never goes down. In fact, it can be argued that if the measurements are viewed as taken from the same group of random numbers distributed according to a Gaussian distribution (often called a Bell Curve), the error of the average should be quoted as reduced by a factor \( 1/\sqrt{N} \) as compared with the above. We will study this in a later lab.

**ESTIMATING ERROR FOR A SMALL AMOUNT OF DATA**

Often in elementary lab courses, only a small amount of data is taken. For example, in the lab studying motion down an incline, we only took 10 time trials at each height. In a case like this, statistical estimations do not make much sense, because they only work with large numbers of data. So often one simply estimates the errors associated with the data by looking at the scatter of the data. Here are two ways people sometimes use in this circumstance. First, you can calculate the average, and then see which data point deviates the farthest from the average. The absolute magnitude of the difference between the average and this data point can be used as a rather conservative error estimate which includes all the data within error bars. Alternatively, perhaps a slightly more realistic estimate would be to use half the difference between the largest and smallest values as an error estimate. This method may put a data point or two outside the error bars, but serves potentially as a slightly tighter estimate of error. (If your error estimate for each data point is substantially smaller than the estimate obtained here, you have probably misjudged how accurately you could make measurements.)
PERCENT ERROR

If there is a “correct” or accepted value $A$ for a quantity you are trying to determine in an experiment, you may wish to compute how closely you came to this accepted value. Let us assume that you have obtained an experimental value $E$ of the quantity in question ($E$ being the average of your individual data points), and you wish to compare with the accepted value $A$. One way of quoting the accuracy of your experiment would be by calculating the absolute difference between the experimental value $E$ and the accepted value $A$, written $|E - A|$, which is the positive difference in the values. (Simply subtract the smaller value from the larger.)

Another way to quote your accuracy is with the fractional error, which is the ratio of the absolute difference to the accepted value:

$$\text{Fractional error} = \frac{\text{absolute difference}}{\text{accepted value}} = \frac{|E - A|}{A}.$$  

The fractional error is commonly expressed as a percentage to give the percent error of an experimental value.

$$\% \text{ error} = \frac{\text{absolute difference}}{\text{accepted value}} \times 100\% = \frac{|E - A|}{A} \times 100\%.$$  

For example, if you were measuring the value of the acceleration due to gravity near the surface of the earth, and you got a value of 9.3 m/s$^2$ and of course the accepted value is 9.8 m/s$^2$, then the percent error is

$$\% \text{error} = \frac{|9.3 - 9.8|}{9.8} \times 100\% = \frac{0.5}{9.8} \times 100\% = 5\%.$$  

PERCENT DIFFERENCE

When you don’t know the accepted value, another quantity sometimes used to compare two experimental quantities is the percent difference. This is just the absolute difference between the two quantities divided by their average value times 100%:

$$\% \text{ difference} = \frac{\text{absolute difference}}{\text{average}} \times 100\% = \frac{|E_1 - E_2|}{\frac{1}{2}(E_1 + E_2)} \times 100\%,$$

where $E_1$ and $E_2$ are two experimental values.